CDO modelling from a practitioner’s point of view: What are the real problems?

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Bridging between academia and practice

- The speaker
- iTraxx, standard CDOs and conventions
- Gaussian copula model
  - CDO behaviour
  - Correlation smile
  - Compound ↔ base correlations
  - Some base correlation issues
- What is a good model?
  - Interpolation and extrapolation, non standard tranches/portfolios
  - Market information
  - Hedge ratios
- Implementation considerations
  - MC strategies, how to simulate
  - Risk numbers for all market data
  - Fast recursive techniques, conditional independence
  - Other model proposals
- What makes a good practitioner?
- Conclusion
Jens Lund

- Head of Product Development, Nordea Markets

- Background:
  - Nov 1996: M.Sc. in statistics, University of Copenhagen
  - Feb 2000: Ph.D. in statistics, The Royal Veterinary and Agricultural University
  - Mar 2000 onwards: with Nordea, Product Development
  - Has done a lot of the credit modelling work in Nordea

- Team:
  - 5 members, various degrees of experience, mainly Ph.D. in natural science, looking for more people
  - 2 associated programmers helping with interface to trading system
  - Responsible for all derivatives modelling and calculations (NPV, risk, ...)
  - Scripting language for description of all derivatives
  - Interest rates, credit derivatives, inflation, equity, ...
iTraxx standard portfolio/CDS

- iTraxx Europe
  - 125 liquid names
  - Underlying index CDSes for sectors
  - 5Y, 7Y & 10Y maturity
  - 5 standard CDO tranches, first to default baskets, options
  - US index CDX

- 3m, act/360, last 20 date roll, CDS pay accrued fee

- Index composition adjusted every 6m

- Index CDS trades at a fixed spread with accrued fee

  Traded with upfront premium (but quoted on spread)

  - Together with last 20 date roll this ensures liquidity and (minus counterparty risk) perfect netting of trades.
iTraxx, distribution of 5y spreads

TDC 2 Nov 2005: 293bp -> 268bp -> 290bp, Mid March at approx 270bp
iTraxx average spreads, 5y mean = 37bp
Average market implied survival probability
Standardized CDO tranches

- iTraxx Europe 3%, 6%, 9%, 12%, 22%
  - US index CDX has points 3%, 7%, 10%, 15%, 30%
- Has done a lot to provide liquidity in structured credit
- Reliable pricing information available
- Quotation:
  - bp running fee
  - Equity tranche:
    - 500bp running, quoted on upfront payment!
  - Due to timing risk of events
Reference Gaussian copula model

- N credit names, i = 1,…,N
- Default times: \( T_i \sim F_i(t) = 1 - \exp\left(-\int_0^t \lambda_i(u)du\right) \)
- \( \lambda_i \) curves bootstrapped from CDS quotes
- \( T_i \) correlated through the copula:
  \[
  F_i(T_i) = \Phi(X_i) \text{ with } X = (X_1,\ldots,X_N)^t \sim N(0,\Sigma)
  \]
  \( \Sigma \) correlation matrix, variance 1, constant correlation \( \rho \)
  Could take
  \[
  X_i = \sqrt{\rho} M + \sqrt{1-\rho} Z_i
  \]
- In model: correlation independent of product to be priced
CDO behaviour

- Structure:
  - 125 name, iTraxx
  - RR almost all 40%
  - Avg CDS = 37bp
  - Corr = 25%
  - Start 11-oct-2005
  - 5y structure, ends 20-dec-2010
  - Premium: 3m, act/360
  - Valuation 10-oct-2005

  Spreads with corr = 25%

  78% Super senior

  0.2bp

  8bp

  87bp

  242bp

  500bp running + 20.55% upfront

  Mezzanine

  3% equity

  3% equity
Correlation dependence
Fair spreads as function of correlation
CDO behaviour depends on

- Number of names
- Spreads of the underlying names
- Tranching:
  - Size of tranche
    - Smaller tranches are more leverage/exposed to changes
  - Order of tranche
- Correlation
- Recovery rate
Prices in the market have a correlation smile

- In practice:
  
  Correlation depends on product, 10-oct-2005, 5Y iTraxx Europe

  - Tranche
  - Maturity
  - Fair coupons

  Equity upfront: 29.2%
  3-6%: 0.97%
  6-9%: 0.28%
  9-12%: 0.13%
  12-22%: 0.07%
Compound correlations

- The correlation on the individual tranches
- Mezzanine tranches have low correlation sensitivity and even non-unique or non-existent correlation for given spreads!
- No way to extend to, say, 2%-5% tranche or bespoke tranches

- What alternatives exists?
Base correlations

![Graph showing base correlations and detachment point with short and long labels](image-url)
Steep base correlation curves

<table>
<thead>
<tr>
<th>Base smile</th>
<th>Corr</th>
<th>Fair coupons</th>
</tr>
</thead>
<tbody>
<tr>
<td>3%</td>
<td>5%</td>
<td>33.94%</td>
</tr>
<tr>
<td>6%</td>
<td>30%</td>
<td>-0.65%</td>
</tr>
<tr>
<td>9%</td>
<td>45%</td>
<td>-0.13%</td>
</tr>
<tr>
<td>12%</td>
<td>50%</td>
<td>0.41%</td>
</tr>
<tr>
<td>22%</td>
<td>55%</td>
<td>0.31%</td>
</tr>
</tbody>
</table>

Negative spreads!!

- Base correlations depends on previous points
- Somewhat contradicting the whole idea of base tranches!
Are base correlations a real solution?

- No, it is merely a convenient way of describing prices on CDO tranches
- An intermediate step towards better models that exhibit a smile
- No general extension to other products
- No smile dynamics
- Interpolation issues
- Correlation smile modelling, versus
- Models with a smile and correlation dynamics
- Base correlation is NOT a model!!
- Nevertheless: they are used a lot!
Why have models?
How to use them?

- We do see prices on the standard tranches in the market, so why have a model at all?
- Interpolation
  - Non standard tranches, e.g. 2%-4%
- Extrapolation
  - Attachment/detachment points below 3%
  - Bespoke portfolios
  - Other products: CDS -> CDO, CDO -> CDO², etc.
- Usually: map expected losses to find corr for other tranches
- Risk numbers
Delta hedges

- Delta risk: how much does the NPV change when the underlying credit spreads widen by 1bp?
- CDO tranches typical traded with initial credit hedge, i.e. only correlation risk left!
- Conveniently quoted as amount of underlying index CDS to buy in order to hedge credit risk, i.e. $\frac{\text{deltaCDO}}{\text{deltaCDS}}$
- Split out on individual names or just consider index?
- Base correlation: find by long/short strategy in the same way as NPV!
Deltas in different models

- Deltas differ between models:

<table>
<thead>
<tr>
<th>Tranche</th>
<th>0%-3%</th>
<th>3%-6%</th>
<th>6%-9%</th>
<th>9%-12%</th>
<th>12%-22%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compound corr</td>
<td>22.1</td>
<td>9.1</td>
<td>2.7</td>
<td>1.2</td>
<td>0.6</td>
</tr>
<tr>
<td>Base corr</td>
<td>22.1</td>
<td>6.1</td>
<td>2.0</td>
<td>0.9</td>
<td>0.5</td>
</tr>
<tr>
<td>RFL</td>
<td>25.9</td>
<td>7.5</td>
<td>0.4</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

- Agreement on delta amounts requires model agreement
- Non-unique deltas when spread & correlation is connected, i.e. in models with smile dynamics
Last 20 schedule and date roll convention

- End date will be 20th of Mar, Jun, Sep or Dec.
- If we have passed any of these dates we roll to the next date, so e.g. end date 21st Jun will roll to 20st Sep, etc.
- First period will be long if we would otherwise get less than 1m to first date in schedule!
- Stub/long period in the beginning.
- Intermediate points are rolled Following.
- Usually in the credit market start and end dates can fall on non-business days.
- Always start protection the day after the trade day, even if a non business day.
Risk ladders

- CDS curve most often bootstrapped from yearly quotes
  - Risk on the yearly quotes, 1Y, 2Y, 3Y, 4Y, 5Y, 6Y, etc.

- However: trades end every quarter

- Risk might move around when crossing 20 Mar, Jun, Sep, Dec

- Example:

<table>
<thead>
<tr>
<th>Date</th>
<th>4Y</th>
<th>4.25Y</th>
<th>4.50Y</th>
<th>4.75Y</th>
<th>5Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>QTR before roll</td>
<td></td>
<td></td>
<td>200</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y before roll</td>
<td>100</td>
<td></td>
<td></td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>QTR after roll</td>
<td>200</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y after roll</td>
<td>150</td>
<td></td>
<td></td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

- Get risk on a quarterly ladder, even though the curve is still bootstrapped from yearly quotes. Be aware how your risk changes on rolls.
Implementation strategies

- Key: efficiency, flexibility and fast + accurate risk!
- Copula type models:
  - Monte Carlo
  - Recursive/FFT techniques
Implementation of Gaussian copula by MC

- Monte Carlo simulation of $X \sim N(0, \Sigma)$
  - Simulate $Y \sim N(0, I)$
  - Find $A$ such that $AA' = \Sigma$
  - $X = AY$

- How to find $A$?
  - Cholesky decomposition
  - Eigenvalue decomposition: $A = P \sqrt{\lambda}$
  - The latter is better, in particular with Sobol sequences

- Simulation:
  - Simulate default time $T_i = F_i^{-1}(\Phi(X_i))$ for all names, and price.
  - Do it, say, 10000 times.
  - Can price any derivative, simple.
Risk numbers in MC pricing

- “Naive” risk numbers: \[ \frac{\partial V}{\partial \lambda_i} = \frac{V(\lambda_i + \varepsilon) - V(\lambda_i)}{\varepsilon} \]

- For credit risk we can exchange differentiation and integration:

\[ V = E[g(\tau)] = \int g(\tau) f(\tau | \lambda_1, ..., \lambda_N) d\tau \]

\[ \frac{\partial}{\partial \varepsilon} V(\lambda_1, ..., \lambda_i + \varepsilon, ..., \lambda_N) \bigg|_{\varepsilon=0} = \int g(\tau) \frac{\partial}{\partial \varepsilon} f(\tau | \lambda_1, ..., \lambda_i + \varepsilon, ..., \lambda_N) \bigg|_{\varepsilon=0} d\tau \]

\[ = \int g(\tau) \frac{\partial}{\partial \varepsilon} \log f(\tau | \lambda_1, ..., \lambda_i + \varepsilon, ..., \lambda_N) \bigg|_{\varepsilon=0} f(\tau) d\tau \]

\[ = E[g(\tau) \frac{\partial}{\partial \varepsilon} \log f(\tau | \lambda_1, ..., \lambda_i + \varepsilon, ..., \lambda_N) \bigg|_{\varepsilon=0}] \]

- Calculate derivative and price in same simulation, but for a different payout function

- Speed up of factor 5x125=625. Also improves stability!
Recursive/FFT implementation

- Write $X \sim N(0, \Sigma)$ as $X_i = aM + \sqrt{1-a^2}Z_i$, $a = \sqrt{\rho}$

- Only viable for derivatives that depend on the number of defaults or the cumulative loss (perhaps discretised if RR or notional are not equal)

- Conditional on $M$:
  - $X_i$'s are independent $\Rightarrow$
    - For given horizon $T$, the default indicators $\{X_i \leq C_i(T)\}$ are independent
  - Calculate distribution of number of defaults recursively in $N = \#\text{names}$
  - Binomial expression

- Find loss distribution by integrating over $M$

- Fast and no MC noise
Recursive build-up of loss distribution

- Conditional on M, given a time horizon t: independence and \( p_i \) is the probability name i has survived up to time t.

\[
p(n \text{ issuers, } k \text{ defaults}) = p(n-1,k)p_n + (1-p_n)p(n-1,k-1)
\]

- Next: integrate over M
What is the survival probability?

- Let \( X_i = aM + \sqrt{1-a^2}Z_i \), with \( X_i \sim H_i \)
- The model matches quantiles: \( F_i(T_i) = H_i(X_i) \)
- This means the conditional survival probabilities are:

\[
P(T_i \geq t \mid M ) = P(F_i^{-1}(H_i(X_i)) \geq t \mid M )
= P(X_i \geq H_i^{-1}(F_i(t)) \mid M )
= P(Z_i \geq \frac{H_i^{-1}(F_i(t)) - aM}{\sqrt{1-a^2}} \mid M )
\]
The search for better copulas has started...

- “Better” means
  - describing the observed prices in the market for iTraxx
  - produces a correlation smile
  - has a reasonable low number of parameters
    - one can have a view on and interpret
  - has a plausible dynamics for the correlation smile
  - constant parameters can be used on a range of
    - tranches / products
    - maturities
    - (portfolios)
  - Efficient pricing and risk numbers

- Often start from Gaussian model described as a 1 factor model
  - Computational efficiency!
    \[ X_i = aM + \sqrt{1-a^2}Z_i \]
What makes a good practitioner?

- Has a lot of common sense
  - Understands the difference between up and down, elbow and head...

- Understands products and markets

- IT knowledge
  - Excel, Visual Basic, system functionality, …, C++
  - Make things operational, streamline repetitive processes, can implement the math so it works, etc.

- Mathematical skills
  - Required to develop models and make efficient implementation
  - Numerical analysis, analysis, algebra, stochastic models, etc.
Conclusion

- Models will have to be developed further
  - Smile description and dynamics
  - Delta amounts and other relevant risk numbers
  - Bespoke tranches
  - Computational efficiency
  - Will have to go through a couple of iterations

- Market and products are changing over time

- Practical challenges
  - Manage market data & information
  - Provide smooth infrastructure for all the numbers/trades/etc.